Problem Set Solution

- Tutor: Nelson Lam (nelson.lam.lcy@gmail.com)
- Prerequisite Knowledge:
  - Frenet Serret Equation
  - Differentiation by Part
  - Orthonormal Basis & Linear Combination
- References:
  - **Lecture Notes** of Dr. LAU & Dr. CHENG
  - Thomas Calculus
- Submission & Rewards
  - Complete at least 2 harder questions during tutorial
    Reward: Any drink from vending machine
  - Complete all unfamiliar non-standard questions within 24 hours at the date of release
    WARNING : **Revising course material is probably a better way to kill time**
    Reward: A free lunch & Respect from TA, Classmates
- All the suggestions and feedback are welcome. Any report of typos is appreciated.

1 Computational Question

Exercise 1.1 ((a) 2018 TDG Quiz 2 — (b) 2020 TDG Quiz 2 — (c), (d), 2023 TDG Quiz 2)**.**

Compute the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$, curvature κ and torsion τ of the space curves below.

(a). $\alpha(\theta) = (a\cos\theta,\ a\sin\theta,\ b\theta)\,,\ \theta \in \mathbb{R}$

(b). $\alpha(\theta) = \left(t - \sin t\cos t,\ \sin^2 t,\ \cos t\right)\,,\ t \in (0, \pi)$

(c). $\alpha(s) = \left(\frac{4}{9}(1+s)^{\frac{3}{2}}, \frac{4}{9}(1-s)^{\frac{3}{2}}, \frac{1}{3}s\right), s \in (-1, 1)$

(d). $\alpha(\theta) = \left(6\cos 2\theta\cos^3\left(\frac{2\theta}{3}\right),\ 6\sin 2\theta\cos^3\left(\frac{2\theta}{3}\right),\ \frac{1}{2}\cos 4\theta - \cos^2 2\theta\right),\ \theta \in \left(0, \frac{\pi}{4}\right)$

Solution (Solution to (a), (b))*.* Demonstrated in tutorial

Solution (Solution to (c)).

Observe that

$$\alpha'(s) = \left(\frac{2}{3}(1+s)^{\frac{1}{2}}, -\frac{2}{3}(1-s)^{\frac{1}{2}}, \frac{1}{3}\right)$$

and

$$\|\alpha'(s)\|^2 = \frac{4}{9}(1+s) + \frac{4}{9}(1-s) + \frac{1}{9} = 1$$

Hence, α is an arc-length parametrization.
Then, the unit tangent vector is

$$\begin{aligned} \mathbf{T}(s) = \alpha'(s) &= \left(\frac{2}{3}(1+s)^{\frac{1}{2}}, -\frac{2}{3}(1-s)^{\frac{1}{2}}, \frac{1}{3}\right) \\ \mathbf{T}'(s) &= \left(\frac{1}{3}(1+s)^{-\frac{1}{2}}, \frac{1}{3}(1-s)^{-\frac{1}{2}}, 0\right) \end{aligned}$$

and

$$\|\mathbf{T}'(s)\| = \sqrt{\frac{1}{9(1+s)} + \frac{1}{9(1-s)}} = \frac{1}{3}\sqrt{\frac{2}{1-s^2}}$$

Therefore, the curvature is

$$\kappa(s) = \|\mathbf{T}'(s)\| = \frac{1}{3}\sqrt{\frac{2}{1-s^2}}$$

and $\kappa(s) > 0$ for all $s \in (-1,1)$
The unit normal vector is given by

$$\mathbf{N}(s) = \frac{1}{\kappa(s)}\mathbf{T}'(s) = \left(\frac{1}{\sqrt{2}}(1-s)^{\frac{1}{2}}, \frac{1}{\sqrt{2}}(1+s)^{\frac{1}{2}}, 0\right)$$

The binormal vector is given by

$$\mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s) = \left(-\frac{1}{3\sqrt{2}}(1+s)^{\frac{1}{2}}, \frac{1}{3\sqrt{2}}(1-s)^{\frac{1}{2}}, \frac{2\sqrt{2}}{3}\right)$$

Differentiating $\mathbf{N}(s)$ with respect to s, we have

$$\mathbf{N}'(s) = \left(-\frac{1}{2\sqrt{2}}(1-s)^{-\frac{1}{2}}, \frac{1}{2\sqrt{2}}(1+s)^{-\frac{1}{2}}, 0\right)$$

and thus the torsion is given by

$$\tau(s) = \langle \mathbf{N}'(s), \mathbf{B}(s)\rangle = \frac{1}{12}\left(\sqrt{\frac{1+s}{1-s}} + \sqrt{\frac{1-s}{1+s}}\right) = \frac{1}{6\sqrt{1-s^2}}.$$

Candidate Performance Report.

Many of you missed the minus sign for $\mathbf{j}$-component of $\alpha'(s)$ carelessly. It leads you getting wrong $\mathbf{T}, \mathbf{N}, \mathbf{B}$ and torsion $\tau(s)$, so the mark deduction will be serious. I try to give you marks for Q1(a) as many as I can, please do more practice on the computational problems for preparation for final exam!

Solution (Solution to (d)).

First, we compute the differentiation

$$\begin{aligned}\frac{d}{d\theta}\left[6\cos 2\theta\cos^3\left(\frac{2\theta}{3}\right)\right] &= -12\cos 2\theta\cos^2\left(\frac{2\theta}{3}\right)\sin\left(\frac{2\theta}{3}\right) - 12\sin 2\theta\cos^3\left(\frac{2\theta}{3}\right)\\ &= -12\cos^2\left(\frac{2\theta}{3}\right)\left[\cos 2\theta\sin\left(\frac{2\theta}{3}\right) + \sin 2\theta\cos\left(\frac{2\theta}{3}\right)\right]\\ &= -12\cos^2\left(\frac{2\theta}{3}\right)\sin\left(\left(\frac{2\theta}{3}\right)+2\theta\right)\\ &= -12\cos^2\left(\frac{2\theta}{3}\right)\sin\left(\frac{8\theta}{3}\right)\end{aligned}$$

Similarly, we also have

$$\frac{d}{d\theta}\left[6\sin 2\theta\cos^3\left(\frac{2\theta}{3}\right)\right] = 12\cos^2\left(\frac{2\theta}{3}\right)\cos\left(\frac{8\theta}{3}\right)$$

Therefore, we have $\alpha'(\theta) = \left(-12\cos^2\left(\frac{2\theta}{3}\right)\sin\left(\frac{8\theta}{3}\right), 12\cos^2\left(\frac{2\theta}{3}\right)\cos\left(\frac{8\theta}{3}\right), 0\right)$ and

$$\|\alpha'(\theta)\|^2 = 144\cos^4\left(\frac{2\theta}{3}\right) \implies \|\alpha'(\theta)\| = 12\cos^2\left(\frac{2\theta}{3}\right)$$

Hence, the unit tangent vector is

$$\mathbf{T}(\theta) = \left(-\sin\left(\frac{8\theta}{3}\right), \cos\left(\frac{8\theta}{3}\right), 0\right)$$

and

$$\mathbf{T}'(\theta) = \left(-\frac{8}{3}\cos\left(\frac{8\theta}{3}\right), -\frac{8}{3}\sin\left(\frac{8\theta}{3}\right), 0\right)$$

Therefore, the curvature is given by

$$\kappa(\theta) = \frac{\|\mathbf{T}'(\theta)\|}{\|\alpha'(\theta)\|} = \frac{\frac{8}{3}}{12\cos^2\frac{2\theta}{3}} = \frac{2}{9\cos^2\frac{2\theta}{3}}$$

Also, the unit normal vector is

$$\mathbf{N}(\theta) = \left(-\cos\left(\frac{8\theta}{3}\right), -\sin\left(\frac{8\theta}{3}\right), 0\right)$$

Observe that $\mathbf{T}(\theta)\times\mathbf{N}(\theta) = (0,0,1)$ and it is a unit vector.
Therefore, the unit binormal vector is $\mathbf{B} = (0,0,1)$ and the torsion is given by

$$\tau = \langle \mathbf{N}'(\theta), \mathbf{B}\rangle = 0.$$

Candidate Performance Report

Almost all of you failed to figure out $\alpha'(\theta)$ in part (b) and many of you did not simplify $\alpha'(\theta)$ first and directly applied differentiation to get $\alpha''(\theta)$. To be honest, it is the simplest way to kill yourself in part (b), not really recommend you to do so. My marking on Q1(b) is harsh and many of you get 0 or 1 point in this part.

Exercise 1.2 (2015 TDG Quiz 2)**.**
Let $\alpha : \mathbb{R} \to \mathbb{R}^3$ be a space curve defined by: $\alpha(t) = (5 + 13\cos t,\ 12 + 5\sin t,\ 13 - 12\sin t)$

(a). Compute the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$, curvature κ and torsion τ

(b). By considering $\dfrac{d}{dt}\|\alpha(t) - v(t)\|^2$, where $v : \mathbb{R} \to \mathbb{R}^3$ is a space curve. Show that α is a circle and find its centre and radius

Solution (Solution to (a).).
To avoid self-hell, we first find arc-length parametrisation of α

$$\begin{aligned}
\alpha(t) &= (5 + 13\cos t,\ 12 + 5\sin t,\ 13 - 12\sin t) \\
\alpha'(t) &= (-13\sin t,\ 5\cos t,\ -12\cos t) \\
\|\alpha'(t)\|^2 &= (-13\cos t)^2 + (-5\sin t)^2 + (12\sin t)^2 \\
\|\alpha'(t)\| &= 13
\end{aligned}$$

Hence $\alpha(s) = \left(5 + 13\cos\dfrac{s}{13}, 12 + 5\sin\dfrac{s}{13}, 13 - 12\sin\dfrac{s}{13}\right)$ is an arc-length parametrisation of α

$$\mathbf{T}(s) = \alpha'(s) = \left(-\sin\frac{s}{13}, \frac{5}{13}\cos\frac{s}{13}, -\frac{12}{13}\cos\frac{s}{13}\right)$$

$$\begin{aligned}
\alpha''(s) &= \left(-\frac{1}{13}\cos\frac{s}{13}, -\frac{5}{169}\sin\frac{s}{13}, \frac{12}{169}\sin\frac{s}{13}\right) \\
\|\alpha''(s)\| &= \sqrt{\frac{1}{169}\sin^2\frac{s}{13} + \frac{5^2+12^2}{169}\cos\frac{s}{13}} = \frac{1}{13} \\
\mathbf{N}(s) = \frac{\alpha''(s)}{\|\alpha''(s)\|} &= \left(-\cos\frac{s}{13}, -\frac{5}{13}\sin\frac{s}{13}, \frac{12}{13}\sin\frac{s}{13}\right)
\end{aligned}$$

$$\begin{aligned}
\mathbf{B}(s) &= \mathbf{T}(s) \times \mathbf{N}(s) \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\frac{s}{13} & \frac{5}{13}\cos\frac{s}{13} & -\frac{12}{13}\cos\frac{s}{13} \\ -\cos\frac{s}{13} & -\frac{5}{13}\sin\frac{s}{13} & \frac{12}{13}\sin\frac{s}{13} \end{vmatrix} \\
&= \left(0, \frac{12}{13}, \frac{5}{13}\right)
\end{aligned}$$

Note that $\kappa(s) = \|\alpha''(s)\| = \dfrac{1}{13}$ and $\tau(s) = \left\langle \dfrac{d\mathbf{N}}{ds}, \mathbf{B} \right\rangle = -\left\langle \dfrac{d\mathbf{B}}{ds}, \mathbf{N} \right\rangle = 0$

Solution (Solution to (b).). (Disclaimer: This is exactly the pastpaper, the d/dt hint is fake)

By observation, consider $v(t) = (5, 12, 13)$ as a constant vector valued function, then

$$\begin{aligned}
\|\alpha(t) - v(t)\| &= \|(5 + 13\cos t,\ 12 + 5\sin t,\ 13 - 12\sin t) - (5, 12, 13)\| \\
&= \sqrt{(13\cos t)^2 + (5\sin t)^2 + (12\sin t)^2} \\
&= \sqrt{169\cos^2 t + 169\sin^2 t} \\
&= 13
\end{aligned}$$

Hence $\alpha(t)$ lies on a sphere with radius 13 and centre $(5, 12, 13)$
Moreover, $\alpha(t)$ has zero torsion, hence it is contained in a plane
So we can conclude that $\alpha(t)$ represents a circle with same radius and centre

2 Warm up Question

Exercise 2.1 (Proposition 2.4.6 & Lecture Notes Exercise 11, 16)**.**

(a). Let $\mathbf{r}(t)$ be a regular parameterized space curve with curvature $\kappa(t) > 0$ for any t, show that $\mathbf{r}$ is a plane curve if and only if its torsion $\tau(t) = 0$ for all t

(b). Let $\mathbf{r}(t)$ be an regular arc length parameterized space curve with curvature $\kappa(s) = \kappa$ is a constant and $\tau(s) = 0$ for any s . Show that $\mathbf{r}(s)$ lies on a circle. What is the radius of the circle ?

Hints: Try to show that $\dfrac{d}{ds}\left(\mathbf{r}(s) + \dfrac{1}{\kappa}\mathbf{N}(s)\right) = 0$

Exercise 2.2 (Lecture Notes Exercise 14)**.**
Let $\mathbf{r}(t)$ be a regular parameterized plane curve with curvature $\kappa(t) > 0$ for any t. Let $\lambda > 0$ be a constant, the parallel curve $\mathbf{r}_\lambda(t)$ is defined by:

$$\mathbf{r}_\lambda(t) \stackrel{\text{def}}{=} \mathbf{r}(t) - \lambda\,\mathbf{N}(t)$$

where $\mathbf{N}(t)$ is the unit normal vector to $\mathbf{r}$ at t. Show that the curvature of $\mathbf{r}_\lambda(t)$ is $\dfrac{\kappa}{1+\lambda\kappa}$

Solution.
Recall that for arc-length parametrised plane curve $\mathbf{r}(s)$, $\mathbf{N}'(s) = -\kappa(s)\,\mathbf{T}(s)$

If the curve is not arc-length parametrised, by chain rule, $\mathbf{N}'(s) = \|\mathbf{r}'(t)\|(-\kappa(s)\,\mathbf{T}(s))$

Now, consider the parallel curve:

$$\begin{aligned}
\mathbf{r}_\lambda(t) &= \mathbf{r}(t) - \lambda\,\mathbf{N}(t)\\
\frac{d}{dt}\mathbf{r}_\lambda(t) &= \frac{d}{dt}(\mathbf{r}(t) - \lambda\,\mathbf{N}(t))\\
\mathbf{r}'_\lambda(t) &= \mathbf{r}'(t) - \lambda\,\mathbf{N}'(t)\\
\mathbf{r}'_\lambda(t) &= \mathbf{r}'(t) - \lambda(-\kappa(s)\|\mathbf{r}'(t)\|)\,\mathbf{T}(t)\\
\mathbf{r}'_\lambda(t) &= \mathbf{r}'(t) - \lambda(-\kappa(s)\|\mathbf{r}'(t)\|)\frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}\\
\mathbf{r}'_\lambda(t) &= (1+\lambda\kappa(s))\,\mathbf{r}'(t)
\end{aligned}$$

On the other hand, the unit tangent vector for $\mathbf{r}_\lambda$ is:

$$\mathbf{T}_\lambda(t) = \frac{\mathbf{r}'_\lambda(t)}{\|\mathbf{r}'_\lambda(t)\|} = \frac{(1+\lambda\kappa(s))\,\mathbf{r}'(t)}{(1+\lambda\kappa(s))\|\mathbf{r}'(t)\|} = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \mathbf{T}(t) \tag{1}$$

Therefore the curvature of $\mathbf{r}_\lambda(t)$ is:

$$\frac{\mathbf{T}'_\lambda(t)}{\|\mathbf{r}'_\lambda(t)\|} = \frac{\|\mathbf{T}'(t)\|}{(1+\lambda\kappa(s))\|\mathbf{r}'(t)\|} = \frac{1}{1+\lambda\kappa(s)}\left(\frac{\|\mathbf{T}'(t))\|}{\|\mathbf{r}'(t)\|}\right) = \frac{\kappa(s)}{1+\lambda\kappa(s)}$$

Exercise 2.3 (2023 TDG Quiz 2)**.** Let $\mathbf{r}(t)$ be a regular parametrized space curve with $\kappa(t) > 0$ for any t. Denote the torsion at $\mathbf{r}(t)$ by $\tau(t)$. Prove that $\mathbf{r}(t)$ is contained in a plane if and only if $\tau(t) = 0$ for any t.
(Hint: A space curve $\mathbf{r}$ *is contained in a plane if there exists a fixed unit vector* $\mathbf{n}$ *such that* $\langle \mathbf{r}, \mathbf{n} \rangle$ *is a constant.)*

Solution (By Michael Cheung).
Suppose $\mathbf{r}$ is a plane curve. For convenience, we may consider the curve's arc length parametrization $\mathbf{r}(s)$. Choose any fixed point x_0 on $\mathbf{r}(s)$, there exists a constant unit normal vector $\mathbf{n}$ such that

$$\langle \mathbf{r}(s) - x_0, \mathbf{n} \rangle = 0, \ \ i.e.\ \langle \mathbf{r}(s), \mathbf{n} \rangle = \langle x_0, \mathbf{n} \rangle = a$$

which is a constant. Notice that

$$\begin{cases} \langle \mathbf{r}'(s), \mathbf{n} \rangle = \langle \mathbf{r}'(s), \mathbf{n} \rangle + \langle \mathbf{r}(s), \mathbf{n}' \rangle = \dfrac{d}{ds} \langle \mathbf{r}(s), \mathbf{n} \rangle = \dfrac{d}{ds} a = 0 \\ \langle \mathbf{r}''(s), \mathbf{n} \rangle = \langle \mathbf{r}''(s), \mathbf{n} \rangle + \langle \mathbf{r}'(s), \mathbf{n}' \rangle = \dfrac{d}{ds} \langle \mathbf{r}'(s), \mathbf{n} \rangle = 0 \end{cases}$$

From the first equation, we have $\langle \mathbf{T}, \mathbf{n} \rangle = \langle \mathbf{r}'(s), \mathbf{n} \rangle = 0$.
From the second equation, we have $\langle \kappa \mathbf{N}, \mathbf{n} \rangle = \kappa \langle \mathbf{N}, \mathbf{n} \rangle = 0$, i.e. $\langle \mathbf{N}, \mathbf{n} \rangle = 0$ as $\kappa > 0$
Since $\mathbf{B}$ is a unit vector, we have $\mathbf{B} = \mathbf{T} \times \mathbf{N} = \pm \mathbf{n}$, which is a constant vector.
Hence $\mathbf{B}' = -\tau \mathbf{N} = \mathbf{0}$, $i.e.\ \tau \equiv 0\ as$ $\mathbf{N}$ is a non-zero vector.

Conversely, WLOG, suppose $\tau(s) = 0$ for any s. Then we have $\mathbf{B}' = -\tau \mathbf{N} = \mathbf{0}$.
Hence $\mathbf{B}$ is a constant vector and $\dfrac{d}{ds} \langle \mathbf{r}, \mathbf{B} \rangle = \langle \mathbf{r}', \mathbf{B} \rangle + \langle \mathbf{r}, \mathbf{B}' \rangle = \langle \mathbf{T}, \mathbf{B} \rangle - \tau \langle \mathbf{r}, \mathbf{N} \rangle = 0$.
Therefore $\langle \mathbf{r}, \mathbf{B} \rangle$ is a constant, which implies $\mathbf{r}$ is a plane curve lying on a plane with normal vector $\mathbf{B}$.

OR

Conversely, WLOG, suppose $\tau(s) \equiv 0$, then $\mathbf{B}' = -\tau \mathbf{N} \equiv \mathbf{0}$.
Hence $\mathbf{B}$ is a constant vector.
Pick arbitrary point $\mathbf{r}(s_0)$ on $\mathbf{r}$. Consider the plane $\mathbf{f}(s) = \langle \mathbf{r}(s) - \mathbf{r}(s_0), \mathbf{B} \rangle$, we have

$$\mathbf{f}'(s) = \langle \mathbf{r}'(s), \mathbf{B} \rangle + \langle \mathbf{r}(s) - \mathbf{r}(s_0), \mathbf{B}' \rangle = \langle \mathbf{T}, \mathbf{B} \rangle = 0$$

Hence $\mathbf{f}(s)$ is a constant, and $\mathbf{f}(s_0) = \langle \mathbf{r}(s_0) - \mathbf{r}(s_0), \mathbf{B} \rangle = 0$, which implies $\mathbf{f}(s) \equiv 0$
Therefore $\mathbf{r}$ is a plane curve.

Candidate Performance Report.
This question is directly copied from Proposition 2.4.6, p.90 on lecture notes. I understand that the DSE syllabus makes you feel like proofs are not important in Mathematics, but this is a *blunder*! If you cannot answer this question, you are not stupid, you are just lazy and lack preparation. Hope to see your well-prepared performance in your final exam.

3 Standard Question

3.1 Lies on Sphere

Exercise 3.1 (Lecture Notes Exercise 17)**.**
Let $\alpha(s)$ be a regular space curve with arc length parameterization. $\mathbf{N}(s)$ and $\mathbf{B}(s)$ are the unit normal and unit binormal to the curve respectively. Let $\kappa(s)$ and $\tau(s)$ be the curvature and torsion of the curve. Suppose $\alpha(s)$ lies on the unit sphere for any s

(a). Prove that $\langle \alpha, \mathbf{N}\rangle = -\dfrac{1}{\kappa}$ for any s

(b). Prove that $\alpha = -\dfrac{1}{\kappa}\mathbf{N} + \dfrac{\kappa'}{\kappa^2\tau}\mathbf{B}$ for any s

Exercise 3.2 (2021 DG AS1, 2020 TDG Final)**.** Assume that $\kappa(s) > 0$, $\tau(s) \neq 0$ and $\kappa'(s) \neq 0$ for all s for a regular curve $\alpha(s)$ parameterised by arc length. Show that α lies on a sphere if and only if:

$$\frac{1}{\kappa(s)^2} + \frac{\kappa'(s)^2}{\kappa(s)^4\tau(s)^2} = \text{constant}$$

Exercise 3.3. Let $\alpha(s)$, where $s \in (a, b)$ be an arc-length parametrized space curve with $\kappa, \tau \neq 0$ everywhere. Show that α lies on the surface of some sphere if and only if

$$\frac{1}{\kappa(s)^2} + \left(\frac{1}{\tau(s)}\left(\frac{1}{\kappa(s)}\right)'\right)^2 = \text{constant}$$

Solution (One Direction: If lies on sphere...).
Suppose $\alpha(s)$ lies on some sphere, then $\exists r \in \mathbb{R}^+$ such that $\|\alpha(s)\|^2 = r^2$

$$\begin{aligned}
\langle \alpha(s), \alpha(s)\rangle &= r^2\\
\frac{d}{ds}\langle \alpha(s), \alpha(s)\rangle &= 0\\
2\langle \alpha'(s), \alpha(s)\rangle &= 0\\
\langle \alpha'(s), \alpha(s)\rangle &= 0 \qquad \ldots\ldots(1)\\
\frac{d}{ds}\langle \alpha'(s), \alpha(s)\rangle &= 0\\
\langle \alpha''(s), \alpha(s)\rangle + \langle \alpha'(s), \alpha'(s)\rangle &= 0\\
\langle \alpha''(s), \alpha(s)\rangle &= -\langle \alpha'(s), \alpha'(s)\rangle\\
\langle \alpha''(s), \alpha(s)\rangle &= -\|\alpha'(s)\|^2\\
\langle \alpha''(s), \alpha(s)\rangle &= -1 \qquad \ldots\ldots(2)\text{(Since } \alpha \text{ is arc-length parametrised)}\\
\frac{d}{ds}\langle \alpha''(s), \alpha(s)\rangle &= 0\\
\langle \alpha'''(s), \alpha(s)\rangle + \langle \alpha''(s), \alpha'(s)\rangle &= 0\\
\langle \alpha'''(s), \alpha(s)\rangle &= -\underbrace{\langle \alpha''(s), \alpha'(s)\rangle}_{\text{will be shown zero}} \qquad \ldots\ldots(3)
\end{aligned}$$

By Frenet Serret Equation, $\mathbf{T}(s) = \alpha'(s)$, hence (1) implies

$$\langle \alpha(s), \mathbf{T}(s)\rangle = 0$$

By Frenet Serret Equation, $\alpha''(s) = \mathbf{T}'(s) = \kappa(s)\mathbf{N}(s)$, hence (2) implies

$$\langle \alpha(s), \mathbf{N}(s)\rangle = -\frac{1}{\kappa(s)}$$

By assumption, $\alpha(s)$ is arc-lenth parametrised, hence $\langle \alpha'(s), \alpha'(s) \rangle = 1$
Differentiating gives $\langle \alpha''(s), \alpha'(s) \rangle = 0$, hence (3) implies

$$\langle \alpha'''(s), \alpha(s) \rangle = 0$$

Now we investigate $\langle \alpha'''(s), \alpha(s) \rangle$, note that : $\dfrac{d}{ds}\alpha''(s) = \dfrac{d}{ds}(\kappa(s)\,\mathbf{N}(s))$

$$\begin{aligned}
\alpha'''(s) &= \frac{d}{ds}(\kappa(s)\,\mathbf{N}(s)) \\
&= \kappa'(s)\,\mathbf{N}(s) + \kappa(s)\,\mathbf{N}'(s) \\
&= \kappa'(s)\,\mathbf{N}(s) + \kappa(s)[-\kappa(s)\,\mathbf{T}(s) + \tau(s)\,\mathbf{B}(s)] \\
&= -\kappa(s)^2\,\mathbf{T}(s) + \kappa'(s)\,\mathbf{N}(s) + \kappa(s)\tau(s)\,\mathbf{B}(s) \\
\langle \alpha'''(s), \alpha(s) \rangle &= \langle -\kappa(s)^2\,\mathbf{T}(s) + \kappa'(s)\,\mathbf{N}(s) + \kappa(s)\tau(s)\,\mathbf{B}(s), \alpha(s) \rangle \\
0 &= -\kappa(s)^2 \langle \mathbf{T}(s), \alpha(s) \rangle + \kappa'(s)\langle \mathbf{N}(s), \alpha(s) \rangle + \kappa(s)\tau(s)\langle \mathbf{B}(s), \alpha(s) \rangle \qquad \ldots\ldots(4)
\end{aligned}$$

Substitute $\underbrace{\langle \alpha(s), \mathbf{T}(s) \rangle = 0}_{\text{consequence of (1)}}$ and $\underbrace{\langle \alpha(s), \mathbf{N}(s) \rangle = -\dfrac{1}{\kappa(s)}}_{\text{consequence of (2)}}$ into (4)

$$\langle \alpha(s), \mathbf{B}(s) \rangle = \frac{\kappa'(s)}{\kappa(s)^2\tau(s)}$$

Finally, since $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ constitutes an orthonormal basis, hence

$$\alpha(s) = \underbrace{\langle \alpha(s), \mathbf{T}(s) \rangle}_{0}\,\mathbf{T}(s) + \underbrace{\langle \alpha(s), \mathbf{N}(s) \rangle}_{-\frac{1}{\kappa(s)}}\,\mathbf{N}(s) + \underbrace{\langle \alpha(s), \mathbf{B}(s) \rangle}_{\frac{\kappa'(s)}{\kappa(s)^2\tau(s)}}\,\mathbf{B}(s)$$

$$\alpha(s) = -\frac{1}{\kappa(s)}\,\mathbf{N}(s) + \frac{\kappa'(s)}{\kappa(s)^2\tau(s)}\,\mathbf{B}(s)$$

Therefore $\alpha(s)$ lies on a sphere of radius r, by $\langle \mathbf{N}, \mathbf{B} \rangle = 0$, we have

$$\begin{aligned}
r^2 &= \langle \alpha(s), \alpha(s) \rangle \\
&= \left\langle -\frac{1}{\kappa(s)}\,\mathbf{N}(s) + \frac{\kappa'(s)}{\kappa(s)^{@}\tau(s)}\,\mathbf{B}(s), -\frac{1}{\kappa(s)}\,\mathbf{N}(s) + \frac{\kappa'(s)}{\kappa(s)^2\tau(s)}\,\mathbf{B}(s) \right\rangle \\
&= \frac{1}{\kappa(s)^2} + \frac{\kappa'(s)^2}{\kappa(s)^4\tau(s)^2} \\
&= \frac{1}{\kappa(s)^2} + \left(\frac{1}{\tau(s)}\right)^2 \left[\frac{d}{ds}\left(\frac{1}{\kappa(s)}\right)\right]^2
\end{aligned}$$

So we also have a corollary: $\dfrac{1}{\kappa(s)^2} + \left(\dfrac{1}{\tau(s)}\left(\dfrac{1}{\kappa(s)}\right)'\right)^2 = \text{constant}$

Solution (Another Direction: If XXX is constant...).

Suppose $\dfrac{1}{\kappa(s)^2} + \left(\dfrac{1}{\tau(s)}\left(\dfrac{1}{\kappa(s)}\right)'\right)^2 = \text{constant}$

Denote $\rho(s) = [\kappa(s)]^{-1}$ and $\lambda(s) = [\tau(s)]^{-1}$

Construct $\beta(s) = \alpha(s) + \rho(s)\,\mathbf{N}(s) + \rho'(s)\lambda(s)\,\mathbf{B}(s)$

We claim that $\beta(s)$ is a constant vector, so we investigate $\beta'(s)$

$$\begin{aligned}\beta'(s) &= \alpha'(s) + \rho'(s)\,\mathbf{N}(s) + \rho(s)\,\mathbf{N}'(s) + (\rho'(s)\lambda(s))'\,\mathbf{B}(s) + \rho'(s)\lambda(s)\,\mathbf{B}'(s)\\ &= \mathbf{T}(s) + \rho'(s)\,\mathbf{N}(s) + \rho(s)[-\kappa(s)\,\mathbf{T}(s) + \tau(s)\,\mathbf{B}(s)] + (\rho'(s)\lambda(s))'\,\mathbf{B}(s) + \rho'(s)\lambda(s)(-\tau(s)\,\mathbf{N}(s))\\ &= (\rho(s)\tau(s) + (\rho'(s)\lambda(s))')\,\mathbf{B}(s)\end{aligned}$$

It suffices to show that $\rho(s)\tau(s) + (\rho'(s)\lambda(s))' = 0$

By assumption, we have $\dfrac{1}{\kappa(s)^2} + \left(\dfrac{1}{\tau(s)}\left(\dfrac{1}{\kappa(s)}\right)'\right)^2 = \text{constant}$

That is: $\rho(s)^2 + (\rho'(s)\lambda(s))^2 = \text{constant}$

$$\begin{aligned}\frac{d}{ds}(\rho(s)^2 + (\rho'(s)\lambda(s))^2) &= 0\\ 2\rho(s)\rho'(s) + 2(\rho'(s)\lambda(s))(\rho'(s)\lambda(s))' &= 0\\ &= \frac{2\rho'}{\lambda}(\rho(s)\lambda(s) + (\rho'(s)\lambda(s))')\end{aligned}$$

Since $\rho'(s) \neq 0$ and $\lambda(s) = 0$, we conclude that $\rho(s)\lambda(s) + (\rho'(s)\lambda(s))' = 0$, for all $s \in \mathbb{R}$
Hence $\beta'(s) = 0$ and $\beta(s) \equiv \mathbf{a}$ for some constant $\mathbf{a} \in \mathbb{R}^3$, thus

$$\begin{aligned}\beta(s) &= \mathbf{a}\\ \alpha(s) + \rho(s)\,\mathbf{N}(s) + \rho'(s)\lambda(s)\,\mathbf{B}(s) &= \mathbf{a}\\ \alpha(s) - \mathbf{a} &= -(\rho(s)\,\mathbf{N}(s) + \rho'(s)\lambda(s)\,\mathbf{B}(s))\\ \|\alpha(s) - \mathbf{a}\|^2 &= \|\rho(s)\,\mathbf{N}(s) + \rho'(s)\lambda(s)\,\mathbf{B}(s)\|^2\\ &= \langle \rho(s)\,\mathbf{N}(s) + \rho'(s)\lambda(s)\,\mathbf{B}(s), \rho(s)\,\mathbf{N}(s) + \rho'(s)\lambda(s)\,\mathbf{B}(s)\rangle\\ &= \underbrace{\rho(s)^2 + (\rho'(s)\lambda(s))^2}_{\text{constant}}\end{aligned}$$

So $\alpha(s)$ lies on a sphere

3.2 Helix

Exercise 3.4 (Lecture Notes Exercise 18)**.**
Let $\alpha(s)$ be a regular space curve with arc length parameterization. $\mathbf{T}(s)$ and $\mathbf{T}(s)$ are the unit tangent vector and unit normal vector respectively. Suppose $\kappa(s) > 0$ for any s and there exists a constant C and a constant unit vector $\mathbf{u}$ such that $\langle \mathbf{T}(s), \mathbf{u} \rangle = C$ for all s

(a). Show that $\mathbf{N}(s)$ and $\mathbf{u}$ are orthogonal for all s

(b). Using (a), show that there exists a constant θ such that $\mathbf{u} = \cos\theta\,\mathbf{T}(s) + \sin\theta\,\mathbf{B}(s)$ for all s

(c). Using (b) and Frenet Serret equations, show that $\dfrac{\tau(s)}{\kappa(s)} = \cot\theta$

Exercise 3.5 (2021 DG AS1)**.** Prove that for a regular curve: $\dfrac{\kappa}{\tau} = \text{constant} \iff \langle \mathbf{T}, \mathbf{u} \rangle = \cos\theta_0$ for some $u \in \mathbb{R}^3$ and $\theta_0 \in [0, 2\pi)$

Solution (Solution to (a))*.*
Notice that $\forall s \in \mathbb{R}$:

$$\begin{aligned}
\langle \mathbf{T}(s), \mathbf{u} \rangle &= C \\
\frac{d}{ds}\langle \mathbf{T}(s), \mathbf{u} \rangle &= \frac{d}{ds}(C) \\
\langle \mathbf{T}(s), \mathbf{0} \rangle + \langle \mathbf{T}'(s), \mathbf{u} \rangle &= 0 \\
\langle \kappa(s)\,\mathbf{N}(s), \mathbf{u} \rangle &= 0 \\
\kappa(s)\langle \mathbf{N}(s), \mathbf{u} \rangle &= 0
\end{aligned}$$

By assumption, $\kappa(s) > 0, \forall s \in \mathbb{R}$, hence $\langle \mathbf{N}(s), \mathbf{u} \rangle = 0$, therefore $\mathbf{N}(s) \perp \mathbf{u}$

Solution (Solution to (b))*.*
Recall that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ constitute an orthonormal basis for $\mathbb{R}^3$, hence:

$$\begin{aligned}
\mathbf{u} &= \langle \mathbf{u}, \mathbf{T}(s) \rangle\,\mathbf{T}(s) + \langle \mathbf{u}, \mathbf{N}(s) \rangle\,\mathbf{N}(s) + \langle \mathbf{u}, \mathbf{B}(s) \rangle\,\mathbf{B}(s) \\
&= \langle \mathbf{u}, \mathbf{T}(s) \rangle\,\mathbf{T}(s) + \langle \mathbf{u}, \mathbf{B}(s) \rangle\,\mathbf{B}(s)
\end{aligned}$$

It remains to prove that $|\langle \mathbf{u}, \mathbf{T}(s) \rangle| \leq 1$ and $|\langle \mathbf{u}, \mathbf{B}(s) \rangle| \leq 1$. To see this, consider:

$$\begin{aligned}
\|\mathbf{u}\| &= 1 \\
\|\mathbf{u}, \mathbf{u}\| &= 1 \\
\Big\langle \langle \mathbf{u}, \mathbf{T}(s) \rangle\,\mathbf{T}(s) + \langle \mathbf{u}, \mathbf{B}(s) \rangle\,\mathbf{B}(s), \langle \mathbf{u}, \mathbf{T}(s) \rangle\,\mathbf{T}(s) + \langle \mathbf{u}, \mathbf{B}(s) \rangle\,\mathbf{B}(s) \Big\rangle &= 1 \\
\langle \mathbf{u}, \mathbf{T}(s) \rangle^2 \langle \mathbf{T}(s)\,\mathbf{T}(s) \rangle + 0 + \langle \mathbf{u}, \mathbf{B}(s) \rangle^2 \langle \mathbf{B}(s), \mathbf{B}(s) \rangle &= 1 \\
\langle \mathbf{u}, \mathbf{T}(s) \rangle^2 + \langle \mathbf{u}, \mathbf{B}(s) \rangle^2 &= 1
\end{aligned}$$

Therefore there exists $\theta(s)$ such that $\langle \mathbf{u}, \mathbf{T}(s) \rangle = \cos\theta(s)$ and $\langle \mathbf{u}, \mathbf{B}(s) \rangle = \sin\theta(s)$
However, by (a), $\langle \mathbf{u}, \mathbf{T}(s) \rangle = C$, which is a constant, hence $\theta(s) = \bar{\theta}$ independent of s
Thus $\mathbf{u} = \cos\theta\,\mathbf{T}(s) + \sin\theta\,\mathbf{B}(s)$

Solution (Solution to (c)).
By (a), we have $\langle \mathbf{N}(s), \mathbf{u} \rangle = 0$, using differentiation by part:

$$\begin{aligned}
\frac{d}{dt}\langle \mathbf{N}(s), \mathbf{u} \rangle &= 0 \\
\langle \mathbf{N}'(s), \mathbf{u} \rangle + \langle \mathbf{N}(s), \mathbf{0} \rangle &= 0 \\
\langle -\kappa(s)\,\mathbf{T}(s) + \tau(s)\,\mathbf{B}(s), \mathbf{u} \rangle &= 0 \\
-\kappa(s)\langle \mathbf{T}(s), \mathbf{u} \rangle + \tau(s)\langle \mathbf{B}(s), \mathbf{u} \rangle &= 0 \\
-\kappa(s)\langle \mathbf{T}(s), \cos\theta\,\mathbf{T}(s) + \sin\theta\,\mathbf{B}(s) \rangle + \tau(s)\langle \mathbf{B}(s), \cos\theta\,\mathbf{T}(s) + \sin\theta\,\mathbf{B}(s) \rangle &= 0 \\
-\kappa(s)\cos\theta\langle \mathbf{T}(s), \mathbf{T}(s) \rangle + \tau(s)\sin\theta\langle \mathbf{B}(s), \mathbf{B}(s) \rangle &= 0 \\
-\kappa(s)\cos\theta + \tau(s)\sin\theta &= 0 \\
\frac{\tau(s)}{\kappa(s)} &= \cot\theta
\end{aligned}$$

Solution (Solution to Exercise 3.5).
Suppose $\dfrac{\kappa(s)}{\tau(s)}$ =constant, construct the following vector:

$$\mathbf{u}(s) = \frac{\kappa}{\tau}\,\mathbf{B}(s) + \mathbf{T}(s)$$

$$\begin{aligned}
\mathbf{u}'(s) &= \frac{\kappa(s)}{\tau(s)}\,\mathbf{B}'(s) + \mathbf{T}'(s) \\
&= \frac{\kappa(s)}{\tau(s)}(-\tau(s)\,\mathbf{N}(s)) + \kappa(s)\,\mathbf{N}(s) \\
&= 0
\end{aligned}$$

Therefore $\mathbf{u}$ is a constant vector. Moreover, $\langle \mathbf{T}(s), \mathbf{u} \rangle = 1$ by construction. Hence proved.

4 Harder Question

Exercise 4.1 (2014 TDG Final Exam, 2019 TDG Quiz 2)**.**
Let $\mathbf{r}(s)$ be a unit speed curve in $\mathbb{R}^3$ with curvature $\kappa > 0$ and torsion τ, prove that the following two statements are equivalent

(1). $\exists$ fixed point $p_0 \in \mathbb{R}^3$, two functions $\lambda(s), \mu(s)$ such that $\mathbf{r}(s) - p_0 = \lambda(s)\,\mathbf{T}(s) + \mu(s)\,\mathbf{B}(s)$

(2). $\dfrac{d}{ds}\left(\dfrac{\tau(s)}{\kappa(s)}\right) = \text{Constant independent of } s$

Solution ((1) $\Rightarrow$ (2)).
Suppose there exists a fixed point $p_0 \in \mathbb{R}^3$ such that $\mathbf{r}(s) - p_0 = \lambda(s)\,\mathbf{T}(s) + \mu(s)\,\mathbf{B}(s)$
Differentiating on both sides with respect to s gives:

$$\begin{aligned}
\frac{d}{ds}\Big(\mathbf{r}(s) - p_0\Big) &= \frac{d}{ds}\Big(\lambda(s)\,\mathbf{T}(s) + \mu(s)\,\mathbf{B}(s)\Big)\\
\mathbf{r}'(s) &= \lambda(s)\,\mathbf{T}'(s) + \lambda'(s)\,\mathbf{T}(s) + \mu(s)\,\mathbf{B}'(s) + \mu'(s)\,\mathbf{B}(s)\\
\mathbf{T}(s) &= \lambda(s)\kappa(s)\,\mathbf{N}(s) + \lambda'(s)\,\mathbf{T}(s) - \mu(s)\tau(s)\,\mathbf{N}(s) + \mu'(s)\,\mathbf{B}(s)\\
\mathbf{T}(s) &= \lambda'(s)\,\mathbf{T}(s) + [\lambda(s)\kappa(s) - \mu(s)\tau(s)]\,\mathbf{N}(s) + \mu'(s)\,\mathbf{B}(s)\\
0 &= (\lambda'(s) - 1)\,\mathbf{T}(s) + [\lambda(s)\kappa(s) - \mu(s)\tau(s)]\,\mathbf{N}(s) + \mu'(s)\,\mathbf{B}(s)
\end{aligned}$$

Since $\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)$ is a orthnormal basis for $\mathbb{R}^3$, they are linearly independent

Hence we have: $\begin{cases} \lambda'(s) - 1 = 0 & \ldots(1) \\ \lambda(s)\kappa(s) - \mu(s)\tau(s) = 0 & \ldots(2) \\ \mu'(s) = 0 & \ldots(3) \end{cases}$

Solving (1), (3), we have $\lambda(s) = s + C, \mu(s) = D$, for some constants $C, D \in \mathbb{R}$
Substitute into (2), we have:

$$\begin{aligned}
(s + C)\kappa(s) - D\tau(s) &= 0\\
\frac{\tau(s)}{\kappa(s)} &= \frac{s}{D} + \frac{C}{D}\\
\frac{d}{ds}\left(\frac{\tau(s)}{\kappa(s)}\right) &= \frac{1}{D}
\end{aligned}$$

Solution ((2) $\Rightarrow$ (1)).
Suppose $\dfrac{d}{ds}\left(\dfrac{\tau(s)}{\kappa(s)}\right) = \text{constant independent of } s$

Then we can write $\dfrac{\tau(s)}{\kappa(s)} = as + b$ for some $a, b \in \mathbb{R}$ independent of s. Then $s + \dfrac{b}{a} = \dfrac{1}{a}\dfrac{\tau(s)}{\kappa(s)}$

$$\begin{aligned}
\text{Consider } & \frac{d}{ds}\left[\mathbf{r}(s) - \left(s + \frac{b}{a}\right)\mathbf{T}(s) - \frac{1}{a}\,\mathbf{B}(s)\right]\\
&= \mathbf{r}'(s) - \left(s + \frac{b}{a}\right)\mathbf{T}'(s) - \mathbf{T}(s) - \frac{1}{a}\,\mathbf{B}'(s)\\
&= \mathbf{T}(s) - \left(s + \frac{b}{a}\right)(\kappa(s)\,\mathbf{N}(s)) - \mathbf{T}(s) + \frac{\tau(s)}{a}\,\mathbf{N}(s)\\
&= [\,\mathbf{T}(s) - \mathbf{T}(s)] - \frac{1}{a}\frac{\tau(s)}{\kappa(s)}(\kappa(s)\,\mathbf{N}(s)) + \frac{\tau(s)}{a}\,\mathbf{N}(s)\\
&= \mathbf{0}
\end{aligned}$$

Hence $\mathbf{r}(s) - \left(s + \dfrac{b}{a}\right)\mathbf{T}(s) - \dfrac{1}{a}\,\mathbf{B}(s) = p_0$ for some $p_0 \in \mathbb{R}^3$, with $\lambda(s) = s + \dfrac{b}{a}$ and $\mu(s) = \dfrac{1}{a}$

5 Unfamiliar Non-Standard Question

Exercise 5.1 (2023 TDG Quiz 2)**.**
Let $\alpha(s) : I \to \mathbb{R}^2$ be a regular arc length parametrized plane curve. Suppose that $\mathbf{p} \in \mathbb{R}^2$ is a point such that $\alpha(s) \neq \mathbf{p}$ for all $s \in I$. Suppose there exists $s_0 \in I$ such that

$$\|\alpha(s_0) - \mathbf{p}\| = \max_{s \in I}\{\|\alpha(s) - \mathbf{p}\|\}.$$

Denote the curvature of α at $s = s_0$ by $\kappa(s_0)$. Show that

$$|\kappa(s_0)| \geq \min_{s \in I}\left\{\frac{1}{\|\alpha(s) - \mathbf{p}\|}\right\}.$$

(**Hint 1**: recall the definition of relative extreme points in differential calculus.)
(**Hint 2**: Any relations between inner product and norm?)

Solution (Credit to Max Shung).
First, note that $\alpha(s) \neq \mathbf{p}$ for all $s \in I$, hence $\|\alpha(s) - \mathbf{p}\|$ is differentiable on I.
Then, we consider

$$\begin{aligned}
\frac{d}{ds}\|\alpha(s) - \mathbf{p}\| &= \frac{\langle \alpha'(s), \alpha(s) - \mathbf{p}\rangle}{\|\alpha(s) - \mathbf{p}\|} \\
\frac{d^2}{ds^2}\|\alpha(s) - \mathbf{p}\| &= \frac{\langle \alpha''(s), \alpha(s) - \mathbf{p}\rangle + \langle \alpha'(s), \alpha'(s)\rangle}{\|\alpha(s) - \mathbf{p}\|} - \frac{\langle \alpha'(s), \alpha(s) - \mathbf{p}\rangle}{\|\alpha(s) - \mathbf{p}\|^2}\frac{d}{ds}\|\alpha(s) - \mathbf{p}\|
\end{aligned}$$

Since $\|\alpha(s) - \mathbf{p}\|$ attains maximum at $s = s_0 \in I$, hence it follows that

$$\left.\frac{d}{ds}\right|_{s=s_0}\|\alpha(s) - \mathbf{p}\| = 0 \implies \frac{\langle \alpha'(s_0), \alpha(s_0) - \mathbf{p}\rangle}{\|\alpha(s_0) - \mathbf{p}\|} = 0$$

and

$$\left.\frac{d^2}{ds^2}\right|_{s=s_0}\|\alpha(s) - \mathbf{p}\| \leq 0 \implies \frac{\langle \alpha''(s_0), \alpha(s_0) - \mathbf{p}\rangle + 1}{\|\alpha(s_0) - \mathbf{p}\|} \leq 0$$

as $\langle \alpha'(s), \alpha'(s)\rangle = 1$ for any $s \in I$.
Observe that

$$\frac{\langle \alpha'(s_0), \alpha(s) - \mathbf{p}\rangle}{\|\alpha(s) - \mathbf{p}\|} = 0, \quad \frac{\alpha(s_0) - \mathbf{p}}{\|\alpha(s_0) - \mathbf{p}\|} = \pm\mathbf{N}(s_0),$$

where $\mathbf{N}(s_0)$ denotes the unit normal vector to α at $s = s_0$.
Therefore, it follows that

$$\begin{aligned}
\big\langle \alpha''(s_0), \pm\mathbf{N}(s_0)\big\rangle + \frac{1}{\|\alpha(s_0) - \mathbf{p}\|} &\leq 0 \\
-\langle \kappa(s_0)\mathbf{N}(s_0), \pm\mathbf{N}(s_0)\rangle &\geq \frac{1}{\|\alpha(s_0) - \mathbf{p}\|} \\
\mp\kappa(s_0) \cdot 1 &\geq \frac{1}{\|\alpha(s_0) - \mathbf{p}\|} \\
|\kappa(s_0)| &\geq \frac{1}{\|\alpha(s_0) - \mathbf{p}\|} = \frac{1}{\max_{s \in I}\|\alpha(s) - \mathbf{p}\|} \qquad \ldots(*)
\end{aligned}$$

As $\alpha(s) \neq \mathbf{p}$ for all $s \in I$, hence $\|\alpha(s) - \mathbf{p}\| > 0$ and it is bounded above by $\|\alpha(s_0) - \mathbf{p}\|$, it follows that

$$\frac{1}{\|\alpha(s) - \mathbf{p}\|} \geq \frac{1}{\|\alpha(s_0) - \mathbf{p}\|} \qquad \forall s \in I$$

Therefore, by definition of minimal element of a set, we have

$$\min_{s\in I}\left\{\frac{1}{\|\alpha(s)-\mathbf{p}\|}\right\}=\frac{1}{\|\alpha(s_0)-\mathbf{p}\|}=\frac{1}{\max_{s\in I}\|\alpha(s)-\mathbf{p}\|}$$

and putting back to (*) and thus the result follows.

Candidate Performance Report

Almost all of you cannot state the differentiablility for norm function, and use first and second differentiation to carry forward. Furthermore, none of you can correctly prove that the relation of the reciprocal of a maximum and the minimum of the reciprocal as shown above. The performance for this question is expected because I am a killer! – *Max Shung*

Exercise 5.2 (2022 DG AS1)**.**
Let $\alpha : (a, b) \to \mathbb{R}^3$ be a regular smooth curve parameterized by arc length so that the curvature $\kappa(s) > 0$ globally. Let $s_0 \in (a, b)$. Show that for $s_3 > s_2 > s_1$ sufficiently close to s_0, $\alpha(s_1), \alpha(s_2), \alpha(s_3)$ do not collinear.

Solution.
Suppose, on the contrary, $\alpha(s_i)$ are collinear for $i = 1, 2, 3$

Then $\exists$ constant vectors $\mathbf{v}, \mathbf{n} \in \mathbb{R}^3$ such that $\langle \alpha(s_i) - \mathbf{v}, \mathbf{n} \rangle = 0$ for $i = 1, 2, 3$

Construct auxiliary function $f(s) = \langle \alpha(s) - \mathbf{v}, \mathbf{n} \rangle$

By Rolle's Theorem, $\exists z_1, z_2 \in \mathbb{R}$ such that

$$s_1 < z_1 < s_2 < z_2 < s_3 \text{ and } f'(z_1) = f'(z_2) = 0$$

By Rolle's Theorem, $\exists \eta \in (z_1, z_2)$ such that $f''(\eta) = 0$, that is

$$\langle \alpha''(\eta), \mathbf{n} \rangle = 0$$

By smoothness of α and Sandwich Theorem, let $s_0 \in (a, b)$, then $\lim_{s_i \to s_0} \alpha''(\eta) = \alpha''(s_0)$

Therefore $\langle \alpha''(s_0), \mathbf{n} \rangle = 0$

However, notice that $\alpha''(s_0) = \kappa(s_0)\,\mathbf{N}(s_0)$, therefore $\kappa(s_0)\langle \mathbf{N}(s_0), \mathbf{n} \rangle = 0$

By assumption, $\alpha(s_i)$ are collinear and $s_i \to s_0$, hence

$$\lim_{s_i \to s_0} \mathbf{N}(s_i) = \mathbf{N}(s_0) = \mathbf{N}(s_0) = \mathbf{n}$$

Combining, we have $\kappa(s_0)\langle \mathbf{n}, \mathbf{n} \rangle = 0$, so we have $\kappa(s_0) = 0$. $(\mathbf{n} \neq \mathbf{0})$

Contradiction, hence in first place, $\alpha(s_i)$ does not collinear